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On interpolation of function spaces by methods defined by means of polygons

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Abstract

We describe the spaces obtained by applying the interpolation methods associated to polygons to N-tuples of weighted L^p-spaces, N-tuples of classical Lorentz spaces and some other N-tuples of function spaces.
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1. Introduction

Interpolation of Banach N-tuples is a question that has been of interest from the beginning of abstract interpolation theory. It was considered for the first time in 1961 by Foiaş and

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Lions [21]. Since then, several authors have investigated extensions of the main interpolation methods to N-tuples. Concerning the extension of the real method, we refer, for example, to the papers of Yoshikawa [28], Sparr [26], Fernandez [19,20] and the paper by Peetre and one of the present authors [15]. The case of the complex method was considered by Favini [18]. More information on these multidimensional methods can be found in the article by Cwikel and Janson [16] and in the monograph by Brudnyĭ and Krugljak [5].

The step from two to several spaces involves considerable difficulties, to the effect that basic results in the classical theory for couples are no longer true in general for N-tuples. For example, the equivalence between J- and K-constructions fails. Duality is another difficult point, because duals of these spaces may fail to be intermediate spaces with respect to the dual N-tuple. However, interpolation methods for N-tuples still have important applications in analysis. For instance, they are useful in the investigation of function spaces with dominating mixed derivatives (see [26,5]). They have a role in the development of the classical theory for couples, as it was shown by Asekritova and Krugljak [1]. Working with function spaces, the multidimensional approach is sometimes even more useful than the usual approach by means of couples, as it is pointed out in the article of Asekritova et al. [2].

In this paper, we deal with J and K interpolation methods introduced by Cobos and Peetre [15], which are similar to the real method but incorporating some geometrical elements. They are defined by means of a convex polygon $\Pi = \overline{P_1 \cdots P_N}$, an interior point (α, β) of Π and a scalar parameter $q \in [1, \infty]$. The Banach spaces of the N-tuple should be thought of as sitting on the vertices P_j of Π.

The motivation of Cobos and Peetre for introducing these methods was to follow a new geometrical approach which, on one hand, closes the gap between the ideas of real and complex interpolation, and on the other hand, it gives a unified point of view for other multidimensional methods. Indeed, if Π is equal to the simplex, these methods give back (the first nontrivial case of) spaces studied by Sparr [26], and if Π is equal to the unit square, we recover spaces considered by Fernandez [19,20]. The resulting theory for J and K methods defined by polygons highlights the geometrical aspects of the classical theory of real interpolation for couples. See, for example, besides [15], the papers by Cobos et al. [14], Cobos and Fernández-Martínez [9,10], Cobos et al. [13], Carro et al. [6], Ericsson [17] and Cobos et al. [12].

Sometimes applying multidimensional methods to an N-tuple one gets spaces that can be also obtained by using the real method repeatedly (see [26,25,13]). But this is not always the case. For example, it was shown in [13], Theorem 2.3, that interpolating the N-tuple of L^∞-spaces with weights $(L^\infty(w_1), \ldots, L^\infty(w_N))$ by the K-method associated to the polygon Π, the point (α, β) and $q = \infty$, then the outcome is $L^\infty(\check{w}_{\alpha,\beta})$, the weighted L^∞-space defined by

$$\check{w}_{\alpha,\beta}(x) = \min\{w_i^{c_i}(x) w_j^{c_j}(x) w_k^{c_k}(x) : \{i, j, k\} \in \mathcal{P}_{\alpha,\beta}\}.$$

Here, $\mathcal{P}_{\alpha,\beta}$ is the set of all those triples $\{i, j, k\}$ such that (α, β) belongs to the triangle with vertices P_i, P_j, P_k and (c_i, c_j, c_k) stands for the barycentric coordinates of (α, β) with respect to P_i, P_j, P_k. They also established a similar formula for N-tuples of weighted L^1-spaces when they are interpolated by the J-method with $q = 1$. In this case the weight turns out to be

$$\hat{w}_{\alpha,\beta}(x) = \max\{w_i^{c_i}(x)w_j^{c_j}(x)w_k^{c_k}(x) : \{i, j, k\} \in \mathcal{P}_{\alpha,\beta}\}.$$

These two formulae are genuine results for N-tuples, since they cannot be derived by using the real method again and again.

The aim of this paper is to continue their research by characterizing the spaces that arise by interpolating N-tuples of L^p-spaces with weights by the K- and J-methods and any $1 \leqslant p = q \leqslant \infty$. As we shall show, in the special case where (α, β) does not lie on any diagonal of Π, the new formulae only involve the functions $\check{w}_{\alpha,\beta}$ and $\hat{w}_{\alpha,\beta}$. They are a new genuine N-tuple results. We also study the case of N-tuples formed by classical Lorentz spaces $\overline{\Lambda^p_\phi} = (\Lambda^p_{\phi_1}, \ldots, \Lambda^p_{\phi_N})$. Here, ϕ_j stands for the fundamental function of the jth space $(j = 1, \ldots, N)$. If (α, β) does not belong to any diagonal of Π, then the resulting spaces are Lorentz spaces with fundamental functions $\check{\phi}_{\alpha,\beta}$ and $\hat{\phi}_{\alpha,\beta}$. These function are defined in the same way that $\check{w}_{\alpha,\beta}$ and $\hat{w}_{\alpha,\beta}$ but replacing w_j by ϕ_j. If the point (α, β) belongs to some diagonal of Π, the outcome are weighted spaces (respectively, Lorentz spaces), but with much more involved weights (respectively, fundamental functions).

The dichotomy between to lie or not to lie in any diagonal opens a new line for further research: to clarify the role of the geometry of the polygon.

We also characterize those N-tuples of weighted L^p-spaces for which the K- and J-spaces coincide. This result is particularly interesting since even on simple N-tuples the J- and K-spaces might not be equal (see, for example, [11, Example 3.4]).

To establish these results, we start by computing the K-functional for N-tuples of Lorentz classes associated to Banach function spaces X_j $(j = 1, \ldots, N)$. The concept of Lorentz class $\Lambda^{p,q}(X)$ has been introduced recently by Cerdá, Coll and one of the present authors [8]. The main advantage of working with these classes is that in the arguments it suffices to deal with characteristic functions. Then, as an application of these results, we determine the spaces obtained by interpolation of N-tuples of weighted L^p-spaces, and N-tuples of classical Lorentz spaces.

The paper is organized as follows. In Section 2, we recall the definitions of J- and K-methods defined by means of polygons and some of their basic properties. We also introduce in there the Lorentz classes. In Section 3, we compute the K-functional for N-tuples of Lorentz classes, and we determine the spaces obtained by applying the K-method to an N-tuple of $\Lambda^{p,q}$-spaces. Section 4 contains the K- and J-results for weighted L^p-tuples. We also study there the coincidence of the J- and K-spaces on N-tuples of weighted L^p-spaces, establishing the necessary and sufficient condition for the equality in terms of the weights. Finally, in Section 5, we determine the spaces obtained from an N-tuple of classical Lorentz spaces.

2. Preliminaries

Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon in the plane $\mathbb{R}^2$. The vertices of Π are $P_j = (x_j, y_j)$ $(j = 1 \cdots N)$. Let $\bar{A} = (A_1, \ldots, A_N)$ be a *Banach N-tuple*, that is, a family of N Banach spaces all of them continuously embedded in a common linear Hausdorff space.

We can imagine each space A_j as sitting on the vertex P_j. By means of the polygon Π we define the following family of norms on $\sum(\bar{A}) = A_1 + \cdots + A_N$:

$$K(t, s; a) = K(t, s; a; \bar{A}) = \inf \left\{ \sum_{j=1}^{N} t^{x_j} s^{y_j} \|a_j\|_{A_j} : a = \sum_{j=1}^{N} a_j,\ a_j \in A_j \right\}, \quad t, s > 0.$$

Similarly on $\Delta(\bar{A}) = A_1 \cap \cdots \cap A_N$ we can consider the family of norms

$$J(t, s; a) = J(t, s; a; \bar{A}) = \max_{1 \leqslant j \leqslant N} \{t^{x_j} s^{y_j} \|a\|_{A_j}\}, \quad t, s > 0.$$

Let now (α, β) be an interior point of Π [$(\alpha, \beta) \in Int\ \Pi$] and let $1 \leqslant q \leqslant \infty$. The *K-space* $\bar{A}_{(\alpha,\beta),q;K}$ is defined as the collection of all $a \in \sum(\bar{A})$ for which the norm

$$\|a\|_{(\alpha,\beta),q;K} = \left(\int_0^\infty \int_0^\infty (t^{-\alpha} s^{-\beta} K(t, s; a))^q \frac{dt}{t} \frac{ds}{s} \right)^{1/q}$$

is finite.

The *J-space* $\bar{A}_{(\alpha,\beta),q;J}$ is formed by all those elements $a \in \sum(\bar{A})$ for which there exists a strongly measurable function $u = u(s, t)$ with values in $\Delta(\bar{A})$ such that

$$a = \int_0^\infty \int_0^\infty u(t, s) \frac{dt}{t} \frac{ds}{s} \tag{1}$$

and

$$\left(\int_0^\infty \int_0^\infty (t^{-\alpha} s^{-\beta} J(t, s; u(t, s)))^q \frac{dt}{t} \frac{ds}{s} \right)^{1/q} < \infty. \tag{2}$$

The norm on $\bar{A}_{(\alpha,\beta),q;J}$ is

$$\|a\|_{(\alpha,\beta),q;J} = \inf \left\{ \left(\int_0^\infty \int_0^\infty (t^{-\alpha} s^{-\beta} J(t, s; u(t, s)))^q \frac{dt}{t} \frac{ds}{s} \right)^{1/q} \right\},$$

where the infimum is taken over all representation u satisfying (1) and (2).

The *real interpolation space* $(A_0, A_1)_{\theta,q}$ (see [4,27]) can be seen as the "limit case" when the polygon degenerates into the segment $[0, 1]$. Recall that given any Banach couple (A_0, A_1) the space $(A_0, A_1)_{\theta,q}$ $(0 < \theta < 1, 1 \leqslant q \leqslant \infty)$ is

$$(A_0, A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \|a\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta} K(t, a))^q \frac{dt}{t} \right)^{1/q} < \infty \right\},$$

where $K(t, a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}$. The space $(A_0, A_1)_{\theta,q}$ can be also described in terms of the J-functional $J(t, a) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}$. It turns out that

$$(A_0, A_1)_{\theta,q} = \left\{ a = \int_0^\infty u(t) \frac{dt}{t} : \left(\int_0^\infty (t^{-\theta} J(t, u(t)))^q \frac{dt}{t} \right)^{1/q} < \infty \right\}$$

and $\|\cdot\|_{\theta,q}$ is equivalent to

$$\|a\|_{\theta,q;J} = \inf\left\{\left(\int_0^\infty (t^{-\theta}J(t,u(t)))^q \frac{dt}{t}\right)^{1/q} : a = \int_0^\infty u(t)\frac{dt}{t}\right\}.$$

In contrast with the case of couples, K- and J-methods for N-tuples ($N \geqslant 3$) do not coincide in general. We only have that $\bar{A}_{(\alpha,\beta),q;J} \hookrightarrow \bar{A}_{(\alpha,\beta),q;K}$, where $\hookrightarrow$ means continuous inclusion.

If $\tilde{\Pi} = \overline{P_{j_1}\cdots P_{j_M}}$ is another convex polygon whose M vertices all belong to Π, we can form the M-subtuple $\tilde{A} = (A_{j_1}, \ldots, A_{j_M})$ by selecting from $\bar{A}$ those spaces sitting on vertices of $\tilde{\Pi}$. We designate by $\tilde{K}(t,s;\cdot;\tilde{A})$ and $\tilde{J}(t,s;\cdot;\tilde{A})$ the K- and J-functional defined by means of $\tilde{\Pi}$ over $\sum(\tilde{A})$ and $\Delta(\tilde{A})$, respectively. For $(\alpha,\beta) \in Int\ \tilde{\Pi}$ and $1 \leqslant q \leqslant \infty$, we denote by $\tilde{A}_{(\alpha,\beta),q;K}$ and $\tilde{A}_{(\alpha,\beta),q;J}$ the interpolation spaces defined by $\tilde{\Pi}$ over $\tilde{A}$. The next result follows easily from inequalities

$$K(t,s;a;\bar{A}) \leqslant \tilde{K}(t,s;a;\tilde{A}) \ \text{ if } \ a \in \sum(\tilde{A}),$$
$$\tilde{J}(t,s;a;\tilde{A}) \leqslant J(t,s;a;\bar{A}) \ \text{ if } \ a \in \Delta(\bar{A}).$$

Lemma 2.1. *Let $(\alpha,\beta) \in Int\ \Pi$ and $1 \leqslant q \leqslant \infty$. Then the following continuous inclusions hold*

$$\bar{A}_{(\alpha,\beta),q;J} \hookrightarrow \bigcap_{\tilde{\Pi}} \tilde{A}_{(\alpha,\beta),q;J} \hookrightarrow \sum_{\tilde{\Pi}} \tilde{A}_{(\alpha,\beta),q;K} \hookrightarrow \bar{A}_{(\alpha,\beta),q;K}$$

where the intersection and the sum are taken over all convex polygons $\tilde{\Pi} \subseteq \Pi$ with $(\alpha,\beta) \in Int\ \tilde{\Pi}$.

If $q = 1$ or $q = \infty$, we have the following result (see [11, Theorem 1.5]).

Lemma 2.2. *If (α,β) lies on some diagonal of Π, then*

$$\bar{A}_{(\alpha,\beta),1;J} \hookrightarrow \bigcap_{\{i,k\}\in\mathcal{D}} (A_i, A_k)_{\theta_{i,k},1} \hookrightarrow \sum_{\{i,k\}\in\mathcal{D}} (A_i, A_k)_{\theta_{i,k},\infty} \hookrightarrow \bar{A}_{(\alpha,\beta),\infty;K}.$$

Here, $\mathcal{D}$ denotes the set of all couples $\{i,k\}$ such that (α,β) belongs to the diagonal joining P_i and P_k, and for $\{i,k\} \in \mathcal{D}$, $\theta_{i,k}$ is the unique number $0 < \theta_{i,k} < 1$ such that $(\alpha,\beta) = (1-\theta_{i,k})P_i + \theta_{i,k}P_k$.

Sometimes, we shall need to work with N-tuples formed by quasi-Banach spaces. In this case, $K(t,s;\cdot)$ and $J(t,s;\cdot)$ are only quasi-norms, and $\bar{A}_{(\alpha,\beta),q;K}$ and $\bar{A}_{(\alpha,\beta),q;J}$ are quasi-Banach spaces. The domain of definition of q can be extended, namely $0 < q \leqslant \infty$.

Let (Ω, υ) be a σ-finite measure space. We denote by $L^0 = L^0(\upsilon)$ the space of all (equivalence classes of) υ-measurable functions on Ω which are finite almost everywhere. If $f \in L^0$, its *distribution function* is defined by $\mu_f(t) = \upsilon(\{x : |f(x)| > t\})$, and its *decreasing rearrangement* by $f^*(s) = \inf\{t : \mu_f(t) \leqslant s\}$.

Subsequently, we shall work with Banach spaces X of (equivalent classes of) measurable functions on Ω with the following properties:

(i) Whenever $g \in L^0$, $f \in X$ and $|g| \leqslant |f|$ then $g \in X$ and $\|g\|_X \leqslant \|f\|_X$ (lattice property).
(ii) If $0 \leqslant f_n(\omega) \uparrow f(\omega)$ a.e. then $\|f_n\|_X \uparrow \|f\|_X$ (Fatou property).

We put $\|f\|_X = \infty$ if $f \notin X$.

If the norm of a Banach space E of measurable functions on Ω is equivalent to a norm satisfying (i) and (ii), then we say that E is a *Banach function spaces*. We shall also deal with *quasi-Banach function spaces* which are defined analogously but with $\|\cdot\|_X$ being only a quasi-norm.

Examples of Banach function spaces are L^p spaces for $1 \leqslant p \leqslant \infty$ and $L^{p,q}$ spaces for $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. Recall that

$$L^{p,q} = \left\{ f : \|f\|_{p,q} = \left(\frac{1}{p} \int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

If $0 < p < \infty$ and $0 < q \leqslant \infty$ then spaces $L^{p,q}$ are quasi-Banach spaces.

Lorentz classes $\Lambda^{p,q}(X)$ (for $1 \leqslant p, q < \infty$) associated to a Banach function space X will be of special interest for us. They are defined (see [8]) by the condition

$$\|f\|_{\Lambda^{p,q}(X)} = \left(\int_0^\infty y^{q-1} \|\chi_{\{|f|>y\}}\|_X^{q/p} \, dy \right)^{1/q} < \infty.$$

The functional $\|\cdot\|_{\Lambda^{p,q}(X)}$ is not a norm in general, but only a quasi-norm. Properties of X yield that $\Lambda^{p,q}(X)$ is a quasi–Banach function space. But if $p > 1$ then $\Lambda^{p,q}(X)$ does is a Banach function space because it can be obtained by real interpolation between X and L^∞. Namely (see [8, Theorem 6])

$$(X, L^\infty)_{\theta,q} = \Lambda^{p,q}(X), \quad \theta = (p-1)/p.$$

In what follows we write $\Lambda^p(X) = \Lambda^{p,p}(X)$. If $p = 1$, a simple computation shows that $\|\chi_A\|_{\Lambda^1(X)} = \|\chi_A\|_X$. Hence $\Lambda^1(\Lambda^1(X)) = \Lambda^1(X)$. Moreover, by Cerdá et al. [8], $\Lambda^1(X) \hookrightarrow X$. In general $\Lambda^1(X)$ is not a normed space (see [8, Theorem 1]) but if $f, g \in L^0$ are nonnegative and disjointly supported functions, then $\chi_{\{f+g>y\}} = \chi_{\{f>y\}} + \chi_{\{g>y\}}$ and so triangle inequality holds for these vectors

$$\|f+g\|_{\Lambda^1(X)} \leqslant \|f\|_{\Lambda^1(X)} + \|g\|_{\Lambda^1(X)}. \tag{3}$$

Next, we give an example where Lorentz classes can be calculated easily. By a *weight function* $w(x)$ we mean any positive v-measurable function on Ω.

Example 2.1. Let $1 \leqslant p, q < \infty$ and let $L^{p,q}(w)$ be the weighted Lorentz space, which is defined by using the measure $w\,dv$ instead of dv. Then

$$\Lambda^{p,q}(L^1(w)) = L^{p,q}(w).$$

Indeed, we have

$$\|f\|^q_{\Lambda^{p,q}(L^1(w))} = \int_0^\infty y^{q-1} \|\chi_{\{|f|>y\}}\|^{q/p}_{L^1(w)}\, dy = \frac{1}{q}\, \|f\|^q_{L^{p,q}(w)}.$$

A Banach function space X is said to be *rearrangement invariant* (or *r.i.* in short) if whenever $f \in X$ and g is equimeasurable with f, then $g \in X$ and $\|g\|_X = \|f\|_X$. The *fundamental function* of the r.i. space X is defined by

$$\phi_X(t) = \|\chi_D\|_X, \quad \text{where } D \subseteq \Omega \text{ with } v(D) = t.$$

An important example of r.i. spaces are the Lorentz spaces. If ϕ is an increasing concave function on $(0, \infty)$ such that $\phi(0+) = 0$, then the *Lorentz space* Λ_ϕ consists of all functions $f \in L^0$ which have a finite norm

$$\|f\|_{\Lambda_\phi} = \int_0^\infty f^*(s)\, d\phi(s)$$

(see [24,22,3]). The fundamental function of Λ_ϕ coincides with ϕ. The space Λ_ϕ is the smallest of all r.i. spaces with fundamental function ϕ. When $\phi(t) = t^{1/p}$, then $\Lambda_\phi = L^{p,1}$. The next example describes the relationship between Lorentz spaces and Lorentz classes.

Example 2.2. We have

$$\Lambda_\phi = \Lambda^1(\Lambda_\phi).$$

Indeed, by Krein et al. [22], p. 111, $\|f\|_{\Lambda_\phi} = \int_0^\infty \phi(\mu_f(s))\, ds$. Whence, equality follows by using that $\phi(\mu_f(s)) = \|\chi_{\{|f|>s\}}\|_{\Lambda_\phi}$.

In a more general way, for any r.i. space X, still holds that $\phi_X(\mu_f(s)) = \|\chi_{\{|f|>s\}}\|_X$. Consequently, if $\phi_X(0+) = 0$, then $\|f\|_{\Lambda^1(X)} = \int_0^\infty \phi_X(\mu_f(s))\, ds = \|f\|_{\Lambda_{\phi_X}}$. That is, $\Lambda^1(X) = \Lambda_{\phi_X}$.

For $0 < r < \infty$, the *r-convexification* $X^{(r)}$ of the Banach function space X is defined by

$$X^{(r)} = \{f \in L^0 : \|f\|_{X^{(r)}} = \||f|^r\|_X^{1/r} < \infty\}.$$

The space $(X^{(r)}, \|\cdot\|_{X^{(r)}})$ is a Banach function space if $r \geqslant 1$ (see [23]). The next result shows that any $\Lambda^{p,q}$-class can be realized as a Λ^q-class over the (p/q)-convexification. It is a consequence of equality

$$\left(\int_0^\infty \|\chi_{\{|f|^p>y\}}\|_X\, dy\right)^{1/p} = \left(p\int_0^\infty y^{p-1}\|\chi_{\{|f|>y\}}\|_X\, dy\right)^{1/p}.$$

Proposition 2.1. *Let* $1 \leqslant p, q < \infty$. *Then*

(a) $\Lambda^p(X) = \Lambda^1(X)^{(p)}$.
(b) $\Lambda^{p,q}(X) = \Lambda^1(X^{(p/q)})^{(q)} = \Lambda^q(X^{(p/q)})$.

Let $\bar{X}$ be an N-tuple of Banach function spaces on (Ω, υ). If $\bar{X}$ is *regular*, meaning that $\Delta(\bar{X})$ is dense in each X_j for $1 \leqslant j \leqslant N$, then we can form the dual Banach N-tuple $\bar{X}' = (X_1', \ldots, X_N')$ and for $(\alpha, \beta) \in Int\ \Pi$, $1 \leqslant q < \infty$ and $1/q + 1/q' = 1$, the following duality formulae hold (see [12, Corollary 3.3 and Theorem 3.4])

$$(\bar{X}_{(\alpha,\beta),q;K})' = \bar{X}'_{(\alpha,\beta),q';J} \quad \text{and} \quad (\bar{X}_{(\alpha,\beta),q;J})' = \bar{X}'_{(\alpha,\beta),q';K}. \tag{4}$$

Subsequently, we write $A \preccurlyeq B$ if $A \leqslant cB$ for some constant $c > 0$ independent of quantities A and B. If $A \preccurlyeq B$ and $A \succcurlyeq B$, then we put $A \simeq B$.

3. The K-functional for N-tuples of Lorentz classes

Throughout this section $\Pi = \overline{P_1 \cdots P_N}$ will be a convex polygon in the plane $\mathbb{R}^2$ with vertices $P_j = (x_j, y_j)$ and $\bar{X} = (X_1, \ldots, X_N)$ will be a Banach N-tuple of function spaces on (Ω, υ). Our aim is to describe the K-functional for the N-tuple of Lorentz classes $\Lambda^{p,q}(\bar{X}) = (\Lambda^{p,q}(X_1), \ldots, \Lambda^{p,q}(X_N))$. We shall need some preliminary results. First of all, note that equality $\|f\|_{X_j} = \||f|\|_{X_j}$ and the lattice property of X_j $(j = 1, \ldots, N)$ imply that $\left(\sum(\bar{X}), K(t, s; \cdot)\right)$ is a lattice.

Lemma 3.1. *If* $f \in \sum(\bar{X})$, *then*

$$K(t, s; f; \bar{X}) = \inf \left\{ \sum_{j=1}^{N} t^{x_j} s^{y_j} \|f_j\|_{X_j} : |f| \leqslant \sum_{j=1}^{N} f_j;\ \ f_j \geqslant 0 \right\}. \tag{5}$$

Proof. Given any functions $f_1, \ldots, f_N$ with $f_j \geqslant 0$ and $|f| \leqslant \sum_{j=1}^{N} f_j$, we have

$$|f| = \sum_{j=1}^{N} \frac{f_j |f|}{\sum_{k=1}^{N} f_j}.$$

Whence,

$$\sum_{j=1}^{N} t^{x_j} s^{y_j} \|f_j\|_{X_j} \geqslant \sum_{j=1}^{N} t^{x_j} s^{y_j} \left\| \frac{f_j |f|}{\sum_{k=1}^{N} f_j} \right\|_{X_j} \geqslant K(t, s; f; \bar{X}).$$

This proves the left-hand side inequality in (5). The converse inequality is clear. □

Lemma 3.2. *Let* $A \subseteq \Omega$ *be a measurable set. Then*

$$K(t, s; \chi_A) \simeq \inf \left\{ \sum_{j=1}^{N} t^{x_j} s^{y_j} \|\chi_{A_j}\|_{X_j} : A = \bigcup_{j=1}^{N} A_j,\ A_j \cap A_k = \emptyset \right.$$

$$\left. \text{if } j \neq k,\ A_j \text{ measurable} \right\}.$$

Proof. Given any $c > 1$, there is a decomposition $\chi_A = \sum_{j=1}^N f_j$ $(f_j \geqslant 0)$ such that $\sum_{j=1}^N t^{x_j} s^{y_j} \|f_j\|_{X_j} \leqslant cK(t, s; \chi_A; \bar{X})$. Let

$$\Lambda_j = \{x \in \Omega : f_j(x) \geqslant \max\{f_1(x), \ldots, f_{j-1}(x), f_{j+1}(x), \ldots, f_N(x)\}\}$$

and define the sets Γ_j by

$$\Gamma_1 = \Lambda_1 \quad \text{and} \quad \Gamma_j = \Lambda_j \Big\backslash \bigcup_{1 \leqslant k < j} \Lambda_k.$$

Obviously $\chi_A = \sum_{j=1}^N \chi_{A \cap \Gamma_j}$ and $\chi_{A \cap \Gamma_j} = f_1 \chi_{\Gamma_j} + \cdots + f_N \chi_{\Gamma_j} \leqslant N f_j$ $(j = 1, \ldots, N)$. Hence

$$\begin{aligned} K(t, s; \chi_A; \bar{X}) &\leqslant \sum_{j=1}^N t^{x_j} s^{y_j} \|\chi_{A \cap \Gamma_j}\|_{X_j} \leqslant N \sum_{j=1}^N t^{x_j} s^{y_j} \|f_j\|_{X_j} \\ &\leqslant cNK(t, s; \chi_A; \bar{X}). \qquad \square \end{aligned}$$

For $0 < r < \infty$, we denote by $\bar{X}^{(r)} = (X_1^{(r)}, \ldots, X_N^{(r)})$ the quasi-Banach N-tuple formed by the r-convexifications $X_j^{(r)}$ of the spaces of $\bar{X}$.

Lemma 3.3. *Let $f \in \sum(\bar{X}^{(r)})$. Then*

$$K(t^r, s^r; |f|^r; \bar{X})^{1/r} \simeq K(t, s; f; \bar{X}^{(r)}).$$

Proof. Given any $c > 1$, find a decomposition $|f| = \sum_{j=1}^N f_j (f_j \geqslant 0)$ such that $\sum_{j=1}^N t^{x_j} s^{y_j} \|f_j\|_{X_j^{(r)}} \leqslant cK(t, s; f; \bar{X}^{(r)})$. Since $|f|^r = \left(\sum_{j=1}^N f_j\right)^r \preccurlyeq \sum_{j=1}^N f_j^r$, using Lemma 3.1 we get

$$\begin{aligned} cK(t, s; f; \bar{X}^{(r)}) &\geqslant \sum_{j=1}^N t^{x_j} s^{y_j} \|f_j\|_{X_j^{(r)}} = \sum_{j=1}^N (t^{rx_j} s^{ry_j} \|f_j^r\|_{X_j})^{1/r} \\ &\succcurlyeq K(t^r, s^r; |f|^r; \bar{X})^{1/r}. \end{aligned}$$

On the other hand, if we start with $|f|^r$ and decompose it in the form $|f|^r = \sum_{j=1}^N f_j$ $(f_j \geqslant 0)$ with $\sum_{j=1}^N t^{rx_j} s^{ry_j} \|f_j\|_{X_j} \leqslant cK(t^r, s^r; |f|^r; \bar{X})$ then, letting $g_j = f_j^{1/r}$, we have $|f| = (\sum_{j=1}^N g_j^r)^{1/r} \preccurlyeq \sum_{j=1}^N g_j$. Whence, using again Lemma 3.1, we derive

$$\begin{aligned} cK(t^r, s^r; |f|^r; \bar{X}) &\geqslant \sum_{j=1}^N t^{rx_j} s^{ry_j} \|f_j\|_{X_j} = \sum_{j=1}^N (t^{x_j} s^{y_j} \|g_j\|_{X_j^{(r)}})^r \\ &\succcurlyeq K(t, s; f; \bar{X}^{(r)})^r. \qquad \square \end{aligned}$$

Now, we are ready to prove the main result of this section.

Theorem 3.1. *The following statements hold*:

(i) *Let* $\Lambda^1(\bar{X}) = (\Lambda^1(X_1), \ldots, \Lambda^1(X_N))$. *Then*

$$K(t,s;f;\Lambda^1(\bar{X})) \simeq \int_0^\infty K(t,s;\chi_{\{|f|>y\}};\bar{X})\,dy.$$

(ii) *Let* $\Lambda^{p,q}(\bar{X}) = (\Lambda^{p,q}(X_1), \ldots, \Lambda^{p,q}(X_N))$ $(1 < p, q < \infty)$. *Then*

$$K(t,s;f;\Lambda^{p,q}(\bar{X})) \simeq \left(\int_0^\infty y^{q-1} K(t^p,s^p;\chi_{\{|f|>y\}};\bar{X})^{q/p}\,dy\right)^{1/q}.$$

Proof. (i) Let f and g be two nonnegative measurable disjointly supported functions. We claim that

$$K(t,s;f+g;\Lambda^1(\bar{X})) \leqslant K(t,s;f;\Lambda^1(\bar{X})) + K(t,s;g;\Lambda^1(\bar{X})).$$

Indeed, take any decompositions $f = \sum_{j=1}^N f_j$, $g = \sum_{j=1}^N g_j$ with f_j and g_j nonnegative. Then f_j and g_j are disjointly supported. By (3), $\|f_j+g_j\|_{\Lambda^1(X_j)} \leqslant \|f_j\|_{\Lambda^1(X_j)} + \|g_j\|_{\Lambda^1(X_j)}$. Thus

$$K(t,s;f+g;\Lambda^1(\bar{X})) \leqslant \sum_{j=1}^N t^{x_j} s^{y_j} (\|f_j\|_{\Lambda^1(X_j)} + \|g_j\|_{\Lambda^1(X_j)})$$

which implies the claim.

Take any $f \in \sum \Lambda^1(\bar{X})$. Since $|f| \leqslant \sum_{k\in\mathbb{Z}} 2^{k+1}\chi_{\{2^k<|f|\leqslant 2^{k+1}\}}$, we have that

$$\begin{aligned} K(t,s;f;\Lambda^1(\bar{X})) &\leqslant \sum_{k\in\mathbb{Z}} 2^{k+1} K(t,s;\chi_{\{2^k<|f|\leqslant 2^{k+1}\}};\Lambda^1(\bar{X})) \\ &\leqslant \sum_{k\in\mathbb{Z}} 2^{k+1} K(t,s;\chi_{\{|f|>2^k\}};\Lambda^1(\bar{X})) \\ &\leqslant 4\int_0^\infty K(t,s;\chi_{\{|f|>y\}};\Lambda^1(\bar{X}))\,dy. \end{aligned}$$

On the other hand, Lemma 3.2 and the fact that $\|\chi_A\|_{\Lambda^1(X_j)} = \|\chi_A\|_{X_j}$ imply that $K(t,s;\chi_A;\Lambda^1(\bar{X})) \simeq K(t,s;\chi_A;\bar{X})$. Whence,

$$K(t,s;f;\Lambda^1(\bar{X})) \precsim \int_0^\infty K(t,s;\chi_{\{|f|>y\}};\bar{X})\,dy.$$

Conversely, given any decomposition $|f| = \sum_{j=1}^N f_j$ $(f_j \geqslant 0)$, since $\chi_{\{|f|>Ny\}} \leqslant \sum_{j=1}^N \chi_{\{f_j>y\}}$, we have by Lemma 3.1

$$\begin{aligned} \sum_{j=1}^N t^{x_j} s^{y_j} \|f_j\|_{\Lambda^1(X_j)} &= \int_0^\infty \sum_{j=1}^N t^{x_j} s^{y_j} \|\chi_{\{f_j>y\}}\|_{X_j}\,dy \\ &\geqslant \frac{1}{N}\int_0^\infty K(t,s;\chi_{\{|f|>y\}};\bar{X})\,dy. \end{aligned}$$

Consequently,

$$K(t,s;f;\Lambda^1(\bar{X})) \simeq \int_0^\infty K(t,s;\chi_{\{|f|>y\}};\bar{X})\,dy.$$

(ii) According to Proposition 2.1, $\Lambda^{p,q}(X_j) = \Lambda^1(X_j^{(p/q)})^{(q)}$. Hence, using Lemma 3.3 and (i), we have that

$$\begin{aligned} K(t,s;f;\Lambda^{p,q}(\bar{X})) &\simeq K(t,s;f;\Lambda^1(\bar{X}^{(p/q)})^{(q)}) \simeq K(t^q,s^q;|f|^q;\Lambda^1(\bar{X}^{(p/q)}))^{1/q} \\ &\simeq \left(\int_0^\infty K(t^q,s^q;\chi_{\{|f|^q>y\}};\bar{X}^{(p/q)})\,dy\right)^{1/q} \\ &\simeq \left(\int_0^\infty y^{q-1}K(t^q,s^q;\chi_{\{|f|>y\}};\bar{X}^{(p/q)})\,dy\right)^{1/q} \\ &\simeq \left(\int_0^\infty y^{q-1}K(t^p,s^p;\chi_{\{|f|>y\}};\bar{X})^{q/p}\,dy\right)^{1/q}. \qquad \Box \end{aligned}$$

Corollary 3.1. *Let* $(\alpha,\beta)\in Int\ \Pi$. *Then*

(i) $\Lambda^1(\bar{X})_{(\alpha,\beta),1;K} = \Lambda^1(\bar{X}_{(\alpha,\beta),1;K})$.
(ii) $\Lambda^{p,q}(\bar{X})_{(\alpha,\beta),q;K} = \Lambda^{p,q}(\bar{X}_{(\alpha,\beta),q/p;K})$ *for* $1<p,q<\infty$.

Proof. We only give the proof of (ii) because the proof of (i) is similar. By Theorem 3.1 and Fubini's theorem, we obtain

$$\begin{aligned} &\|f\|^q_{\Lambda^{p,q}(\bar{X})_{(\alpha,\beta),q;K}} \\ &\simeq \int_0^\infty y^{q-1}\int_0^\infty\int_0^\infty t^{-\alpha q}s^{-\beta q}K(t^p,s^p;\chi_{\{|f|>y\}};\bar{X})^{q/p}\,\frac{dt}{t}\,\frac{ds}{s}\,dy \\ &\simeq \int_0^\infty y^{q-1}\int_0^\infty\int_0^\infty (z^{-\alpha}u^{-\beta}K(z,u;\chi_{\{|f|>y\}};\bar{X}))^{q/p}\,\frac{dz}{z}\,\frac{du}{u}\,dy \\ &= \int_0^\infty y^{q-1}\|\chi_{\{|f|>y\}}\|^{q/p}_{\bar{X}_{(\alpha,\beta),q/p;K}}\,dy = \|f\|^q_{\Lambda^{p,q}(\bar{X}_{(\alpha,\beta),q/p;K})}. \qquad \Box \end{aligned}$$

4. Interpolation of N-tuples of weighted L^p-spaces

We assume again that (Ω,ν) is a σ-finite measure space, that $\Pi = \overline{P_1\cdots P_N}$ is a convex polygon in $\mathbb{R}^2$ with vertices $P_j=(x_j,y_j)$ and $(\alpha,\beta)\in Int\ \Pi$.

Definition 4.1. Given any weight functions $w_1,\ldots,w_N$ on Ω we put

$$\hat{w}_{\alpha,\beta}(x) = \inf_{t>0,s>0}\left[\max_{1\leqslant j\leqslant N}\{t^{x_j-\alpha}s^{y_j-\beta}w_j(x)\}\right],$$
$$\check{w}_{\alpha,\beta}(x) = \sup_{t>0,s>0}\left[\min_{1\leqslant j\leqslant N}\{t^{x_j-\alpha}s^{y_j-\beta}w_j(x)\}\right].$$

Let $\mathcal{P}_{\alpha,\beta}$ be the set of all triples $\{i, j, k\}$ such that (α, β) belongs to the triangle with vertices P_i, P_j, P_k. Then, by Cobos et al. [13], Lemma 2.2,

$$\hat{w}_{\alpha,\beta}(x) = \max\{w_i^{c_i}(x)w_j^{c_j}(x)w_k^{c_k}(x) : \{i, j, k\} \in \mathcal{P}_{\alpha,\beta}\}$$

and

$$\check{w}_{\alpha,\beta}(x) = \min\{w_i^{c_i}(x)w_j^{c_j}(x)w_k^{c_k}(x) : \{i, j, k\} \in \mathcal{P}_{\alpha,\beta}\},$$

where (c_i, c_j, c_k) stands for the barycentric coordinates of (α, β) with respect to P_i, P_j, P_k.

If $1 \leqslant p < \infty$ and w is a weight function on Ω, we denote by $L^p(w)$ the *weighted* L^p*-space* formed by all v-measurable functions *f* such that

$$\|f\|_{L^p(w)} = \left(\int_\Omega |f(x)|^p w(x)\, dv(x)\right)^{1/p} < \infty.$$

If $p = \infty$, in order to give a role to the weight w, we define $L^\infty(w)$ as the collection of all v-measurable functions *f* having a finite norm

$$\|f\|_{L^\infty(w)} = \|fw\|_{L^\infty}.$$

It was shown in [13], Theorem 2.3, that

$$(L^\infty(w_1), \ldots, L^\infty(w_N))_{(\alpha,\beta),\infty;K} = L^\infty(\check{w}_{\alpha,\beta}). \tag{6}$$

Next we establish interpolation formulae for N-tuples $\overline{L^p(w)} = (L^p(w_1), \ldots, L^p(w_N))$ for any $p < \infty$. We start by computing the K-functional. For this aim, according to Example 2.1 and Theorem 3.1, it suffices to calculate $K(t, s; \chi_A; \overline{L^1(w)})$. This is done in the following lemma.

Lemma 4.1. *Let $A \subset \Omega$ be a measurable set, then*

$$K(t, s; \chi_A; \overline{L^1(w)}) \simeq \int_A \min_{1 \leqslant j \leqslant N}\{t^{x_j} s^{y_j} w_j(x)\}\, dv(x).$$

Proof. First we check, proceeding as in [26], that without loss of generality we may assume that the weights w_j are discrete valued. Indeed, given any $\varepsilon > 0$, define $\bar{w}_j$ by

$$\bar{w}_j(x) = (1+\varepsilon)^m \quad \text{if} \quad (1+\varepsilon)^{m-1} < w_j(x) \leqslant (1+\varepsilon)^m, \quad m \in \mathbb{Z},$$

then, for all $f \in L^1(w_j)$, we have $\|f\|_{L^1(w_j)} \leqslant \|f\|_{L^1(\bar{w}_j)} \leqslant (1+\varepsilon)\|f\|_{L^1(w_j)}$.

Now we can split the set A in pairwise disjoint measurable sets $\{E_s\}_{s\in\mathbb{Z}}$ such that each weight w_j takes a constant value ϖ_j^s in each E_s. So $A = \bigcup_{s\in\mathbb{Z}} E_s$, and $\chi_A = \sum_{s\in\mathbb{Z}} \chi_{E_s}$. If $\min_{1\leqslant j\leqslant N}\{t^{x_j} s^{y_j} \varpi_j^s\} = t^{x_{j_0}} s^{y_{j_0}} \varpi_{j_0}^s$, for each $x \in E_s$ set $f_{j_0}(x) = 1$ and $f_j(x) = 0$ if $j \neq j_0$. Then $\chi_A = \sum_{j=1}^N f_j$ and

$$\begin{aligned} K(t, s; \chi_A; \overline{L^1(w)}) &\leqslant \sum_{j=1}^N t^{x_j} s^{y_j} \|f_j\|_{L^1(w_j)} = \sum_{s\in\mathbb{Z}} \int_{E_s} \min_{1\leqslant j\leqslant N}\{t^{x_j} s^{y_j} \varpi_j^s\}\, dv \\ &= \int_A \min_{1\leqslant j\leqslant N}\{t^{x_j} s^{y_j} \omega_j(x)\}\, dv(x). \end{aligned}$$

The converse inequality is clear. □

Theorem 4.1. *Let* $\overline{L^p(w)} = (L^p(w_1), \ldots, L^p(w_N))$ *and* $1 \leqslant p < \infty$. *For* $(\alpha, \beta) \in Int\ \Pi$ *set*

$$\pi_{\alpha,\beta}(x) = \int_0^\infty \int_0^\infty \min_{1\leqslant j\leqslant N} \{t^{x_j-\alpha} s^{y_j-\beta} w_j(x)\} \frac{dt}{t} \frac{ds}{s}.$$

Then

(i) $\overline{L^p(w)}_{(\alpha,\beta),p;K} = L^p(\pi_{\alpha,\beta})$.
(ii) *If* (α, β) *does not lie on any diagonal of* Π, *then* $L^p(\pi_{\alpha,\beta}) = L^p(\check{w}_{\alpha,\beta})$.

Proof. (i) Since $L^p(w_j) = \Lambda^p(L^1(w_j))$, Corollary 3.1 yields that

$$\overline{L^p(w)}_{(\alpha,\beta),p;K} = \Lambda^p(\overline{L^1(w)}_{(\alpha,\beta),1;K}).$$

By Lemma 4.1 and Fubini's theorem, we get

$$\begin{aligned}
&\|\chi_{\{|f|>y\}}\|_{\overline{L^1(w)}_{(\alpha,\beta),1;K}} \\
&\simeq \int_{\{|f|>y\}} \int_0^\infty \int_0^\infty \min_{1\leqslant j\leqslant N} \{t^{x_j-\alpha} s^{y_j-\beta} w_j(x)\} \frac{dt}{t} \frac{ds}{s}\, dv(x) \\
&= \int_{\{|f|>y\}} \pi_{\alpha,\beta}(x)\, dv(x).
\end{aligned}$$

Therefore

$$\begin{aligned}
\|f\|^p_{\Lambda^p(\overline{L^1(w)}_{(\alpha,\beta),1;K})} &\simeq \int_0^\infty y^{p-1} \|\chi_{\{|f|>y\}}\|_{\overline{L^1(w)}_{(\alpha,\beta),1;K}}\, dy \\
&= \int_0^\infty y^{p-1} \left(\int_{\{|f|>y\}} \pi_{\alpha,\beta}(x)\, dv(x) \right) dy = \frac{1}{p} \|f\|^p_{L^p(\pi_{\alpha,\beta})}.
\end{aligned}$$

(ii) For any $\lambda, \mu > 0$, we have

$$\begin{aligned}
\pi_{\alpha,\beta}(x) &= \int_0^\infty \int_0^\infty \min_{1\leqslant j\leqslant N} \left\{ \left(\frac{t}{\lambda}\right)^{x_j-\alpha} \left(\frac{s}{\mu}\right)^{y_j-\beta} \lambda^{x_j-\alpha} \mu^{y_j-\beta} w_j(x) \right\} \frac{dt}{t} \frac{ds}{s} \\
&\geqslant \min_{1\leqslant j\leqslant N} \{\lambda^{x_j-\alpha} \mu^{y_j-\beta} w_j(x)\} \int_0^\infty \int_0^\infty \left(\frac{t}{\lambda}\right)^{x_j-\alpha} \left(\frac{s}{\mu}\right)^{y_j-\beta} \frac{dt}{t} \frac{ds}{s}.
\end{aligned}$$

The last integral is finite by Cobos and Peetre [15, p. 374]. Therefore, we derive

$$\check{w}_{\alpha,\beta}(x) = \sup_{\lambda,\mu>0} \left[\min_{1\leqslant j\leqslant N} \{\lambda^{x_j-\alpha} \mu^{y_j-\beta} w_j(x)\} \right] \preccurlyeq \pi_{\alpha,\beta}(x).$$

Similarly, one can check that $\pi_{\alpha,\beta} \preccurlyeq \hat{w}_{\alpha,\beta}$. So

$$L^p(\hat{w}_{\alpha,\beta}) \hookrightarrow \overline{L^p(w)}_{(\alpha,\beta),p;K} = L^p(\pi_{\alpha,\beta}) \hookrightarrow L^p(\check{w}_{\alpha\beta}). \tag{7}$$

Note that no assumption on (α, β) has been used to established (7).

Now, if (α, β) does not lie on any diagonal of Π, then for any $\{i, j, k\} \in \mathcal{P}_{\alpha,\beta}$ the point (α, β) is an interior point of the triangle $\{P_i, P_j, P_k\}$ and so the interpolation space $(L^p(w_i), L^p(w_j), L^p(w_k))_{(\alpha,\beta),p;K}$ is well defined. According to Lemma 2.1 we get

$$\sum_{\{i,j,k\}\in\mathcal{P}_{\alpha,\beta}} (L^p(w_i), L^p(w_j), L^p(w_k))_{(\alpha,\beta),p;K} \hookrightarrow \overline{L^p(w)}_{(\alpha,\beta),p;K}.$$

But for triples of weighted L^p-spaces we know that (see [26, Theorem 8.1])

$$(L^p(w_i), L^p(w_j), L^p(w_k))_{(\alpha,\beta),p;K} = L^p(w_i^{c_i} w_j^{c_j} w_k^{c_k}),$$

where (c_i, c_j, c_k) denote the barycentric coordinates of (α, β) with respect to P_i, P_j, P_k. Consequently,

$$\sum_{\{i,j,k\}\in\mathcal{P}_{\alpha,\beta}} L^p(w_i^{c_i} w_j^{c_j} w_k^{c_k}) = L^p(\check{w}_{\alpha,\beta}) \hookrightarrow \overline{L^p(w)}_{(\alpha,\beta),p;K} = L^p(\pi_{\alpha,\beta}).$$

This completes the proof. □

Remark 4.1. If (α, β) lies on some diagonal of Π, then statement (ii) in Theorem 4.1 is not true in general. For example, let Π be the unit square $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$, let $\alpha = \beta = \frac{1}{2}$, and for $n \in \mathbb{N}$ put

$$w_1(n) = w_4(n) = \frac{1}{\sqrt{n}}, \quad w_2(n) = w_3(n) = \frac{1}{n}.$$

According to Cobos et al. [13], Example 2.8,

$$\begin{aligned}\pi_{1/2,1/2}(n) &= \int_0^\infty \int_0^\infty \min\left\{\frac{t^{-1/2}s^{-1/2}}{\sqrt{n}}, \frac{t^{1/2}s^{-1/2}}{n}, \frac{t^{-1/2}s^{1/2}}{n}, \frac{t^{1/2}s^{1/2}}{\sqrt{n}}\right\} \frac{dt}{t}\frac{ds}{s} \\ &\simeq \frac{\log n}{n}.\end{aligned}$$

Hence

$$(\ell^p(w_1), \ell^p(w_2), \ell^p(w_3), \ell^p(w_4))_{(1/2,1/2),p;K} = \ell^p\left(\frac{\log n}{n}\right).$$

But

$$\check{w}_{1/2,1/2}(n) = \min\left(\sqrt{\frac{1}{\sqrt{n}}\frac{1}{\sqrt{n}}}, \sqrt{\frac{1}{n}\frac{1}{n}}\right) = \frac{1}{n}.$$

Remark 4.2. Corollary 3.1 and Lemma 4.1 allow to describe other interpolation spaces. For example, for an N-tuple of weighted $L^{p,q}$-spaces $(1 < p, q < \infty)$ one can show that

$\overline{L^{p,q}(w)}_{(\alpha,\beta),q;K}$ coincides with the space

$$\left\{ f : \left(\int_0^\infty \int_0^\infty \int_0^\infty y^{q-1} \left(\int_{\{|f|>y\}} \min_{1\leqslant j\leqslant N} \{ t^{x_j-\alpha} s^{y_j-\beta} w_j(x) \} \, dv(x) \right)^{q/p} dy \times \frac{dt}{t} \frac{ds}{s} \right)^{1/q} < \infty \right\}.$$

Next we turn our attention to J-interpolation formulae. The relevant weight is now $\hat{w}_{\alpha,\beta}$. It was established in [13, Theorem 2.5], that

$$(L^1(w_1), \ldots, L^1(w_N))_{(\alpha,\beta),1;J} = L_1(\hat{w}_{\alpha,\beta}). \tag{8}$$

The following result describes the spaces obtained from weighted L^p-spaces for any $p > 1$.

Theorem 4.2. *Let* $\overline{L^p(w)} = (L^p(w_1), \ldots, L^p(w_N))$, $(\alpha, \beta) \in Int\, \Pi$, *let* $1 < p \leqslant \infty$ *and* $1/p + 1/p' = 1$. *Then*

$$\overline{L^p(w)}_{(\alpha,\beta),p;J} = L^p(\sigma_{\alpha,\beta}),$$

where

$$\sigma_{\alpha,\beta}(x) = \left(\int_0^\infty \int_0^\infty \min_{1\leqslant j\leqslant N} \left\{ \frac{t^{x_j-\alpha} s^{y_j-\beta}}{w_j(x)^{p'/p}} \right\} \frac{dt}{t} \frac{ds}{s} \right)^{-p/p'} \quad \text{if } p < \infty$$

and

$$\sigma_{\alpha,\beta}(x) = \left(\int_0^\infty \int_0^\infty \min_{1\leqslant j\leqslant N} \left\{ \frac{t^{x_j-\alpha} s^{y_j-\beta}}{w_j(x)} \right\} \frac{dt}{t} \frac{ds}{s} \right)^{-1} \quad \text{if } p = \infty.$$

Moreover, if (α, β) *does not lie on any diagonal of* Π, *then*

$$\overline{L^p(w)}_{(\alpha,\beta),p;J} = L^p(\hat{w}_{\alpha,\beta}). \tag{9}$$

Proof. Recall that for $1\leqslant r < \infty$ the dual space $(L^r(w))'$ of $(L^r(w))$ is $L^{r'}(w^{-r'/r})$. Hence, using (4),

$$\overline{L^p(w)}_{(\alpha,\beta),p;J} = ((L^{p'}(w_1^{-p'/p}), \ldots, L^{p'}(w_N^{-p'/p}))_{(\alpha,\beta),p';K})'.$$

According to Theorem 4.1/(i), we have $(L^{p'}(w_1^{-p'/p}), \ldots, L^{p'}(w_N^{-p'/p}))_{(\alpha,\beta),p';K} = L^{p'}(\pi_{\alpha,\beta})$ with

$$\pi_{\alpha,\beta}(x) = \int_0^\infty \int_0^\infty \min_{1\leqslant j\leqslant N} \{ t^{x_j-\alpha} s^{y_j-\beta} w_j^{-p'/p}(x) \} \frac{dt}{t} \frac{ds}{s}.$$

Therefore,

$$\overline{L^p(w)}_{(\alpha,\beta),p;J} = (L^{p'}(\pi_{\alpha,\beta}))' = L^p(\sigma_{\alpha,\beta}).$$

The proof of (9) is similar, but using now Theorem 4.1/(ii). □

We end this section by characterizing those N-tuples of weighted L^p-spaces for which the K- and J-methods coincide.

Theorem 4.3. *The following are equivalent.*

(i) *For any* $(\alpha,\beta)\in Int\,\Pi$ *and for any* $1\leqslant p\leqslant\infty$, *we have* $\overline{L^p(w)}_{(\alpha,\beta),p;J} = \overline{L^p(w)}_{(\alpha,\beta),p;K}$.

(ii) *There is* $(\alpha_0,\beta_0)\in Int\,\Pi$ *which does not lie on any diagonal of* Π *and* $1\leqslant p_0\leqslant\infty$ *such that* $\overline{L^{p_0}(w)}_{(\alpha_0,\beta_0),p_0;J} = \overline{L^{p_0}(w)}_{(\alpha_0,\beta_0),p_0;K}$.

(iii) *There are weights* v_1, v_2, v_3 *in* Ω *such that for any* $1\leqslant j\leqslant N$ *we have* $w_j \simeq v_1^{1-x_j-y_j} v_2^{x_j} v_3^{y_j}$.

Proof. Obviously (i) $\Rightarrow$ (ii). To show that (ii) $\Rightarrow$ (iii) choose three vertices P_{i_0}, P_{j_0} and P_{k_0} in Ω such that (α_0,β_0) belongs to interior of the triangle $\{P_{i_0}, P_{j_0}, P_{k_0}\}$. Since two polygons related by an affine isomorphism generate the same interpolation space (see [14, Remark 4.1]) we can assume without loss of generality that $P_{i_0}=(0,0)$, $P_{j_0}=(1,0)$, $P_{k_0}=(0,1)$. Put $v_1=w_{i_0}$, $v_2=w_{j_0}$ and $v_3=w_{k_0}$.

According to Theorems 4.1, 4.2, and (6) and (8), we have

$$L^{p_0}(\hat{w}_{\alpha_0,\beta_0}) = \overline{L^{p_0}(w)}_{(\alpha_0,\beta_0),p_0;J} = \overline{L^{p_0}(w)}_{(\alpha_0,\beta_0),p_0;K} = L^{p_0}(\check{w}_{\alpha_0,\beta_0}).$$

Whence, for any triple $\{i,j,k\}\in\mathcal{P}_{\alpha_0,\beta_0}$, since $\check{w}_{\alpha_0,\beta_0}\leqslant w_i^{c_i}w_j^{c_j}w_k^{c_k}\leqslant\hat{w}_{\alpha_0,\beta_0}$, we get

$$L^{p_0}(\hat{w}_{\alpha_0,\beta_0}) = L^{p_0}(w_i^{c_i}w_j^{c_j}w_k^{c_k}) = L^{p_0}(\check{w}_{\alpha_0,\beta_0}).$$

This implies that $\hat{w}_{\alpha_0,\beta_0}\simeq w_i^{c_i}w_j^{c_j}w_k^{c_k}\simeq\check{w}_{\alpha_0,\beta_0}$, and so

$$w_i^{c_i}w_j^{c_j}w_k^{c_k}\simeq v_1^{1-\alpha_0-\beta_0}v_2^{\alpha_0}v_3^{\beta_0}. \tag{10}$$

Take any $r=1,\ldots,N$ with $r\neq i_0, j_0, k_0$. Since (α_0,β_0) does not lie on any diagonal of Π, the point (α_0,β_0) should be in the interior of one of the following triangles $\{P_r, P_{j_0}, P_{k_0}\}$, $\{P_{i_0}, P_r, P_{k_0}\}$, $\{P_{i_0}, P_{j_0}, P_r\}$. Let us assume, for example, that $(\alpha_0,\beta_0)\in Int\,\{P_{i_0}, P_{j_0}, P_r\}$. Since the barycentric coordinates of (α_0,β_0) with respect to P_{i_0}, P_{j_0}, P_r are

$$\left(1-\alpha_0+\frac{\beta_0 x_r}{y_r}-\frac{\beta_0}{y_r},\ \alpha_0-\frac{\beta_0 x_r}{y_r},\ \frac{\beta_0}{y_r}\right),$$

it follows from (10) that $w_r\simeq v_1^{1-x_r-y_r}v_2^{x_r}v_3^{y_r}$. The cases $(\alpha_0,\beta_0)\in\{P_r, P_{j_0}, P_{k_0}\}$ and $(\alpha_0,\beta_0)\in\{P_{i_0}, P_r, P_{k_0}\}$ are analogous.

To complete the proof we will show that (iii) $\Rightarrow$ (i). Since $w_j\simeq v_1^{1-x_j-y_j}v_2^{x_j}v_3^{y_j}$ $(1\leqslant j\leqslant N)$, for any $(\alpha,\beta)\in Int\,\Pi$, we get $\hat{w}_{\alpha,\beta}\simeq\check{w}_{\alpha,\beta}\simeq v_1^{1-\alpha-\beta}v_2^{\alpha}v_3^{\beta}$. By (7) and (6), we have that $\overline{L^p(w)}_{(\alpha,\beta),p;K} = L^p(\hat{w}_{\alpha,\beta})$ for any $1 \leqslant p \leqslant \infty$. Whence, using (8), we obtain for $p = 1$,

$$\overline{L^1(w)}_{(\alpha,\beta),1;J} = L^1(\hat{w}_{\alpha,\beta}) = \overline{L^1(w)}_{(\alpha,\beta),1;K}.$$

If $1 < p \leqslant \infty$, working with the weights

$$u_j = w_j^{-p'/p} \simeq (v_1^{-p'/p})^{1-x_j-y_j}(v_2^{-p'/p})^{x_j}(v_3^{-p'/p})^{y_j},$$

we derive that

$$(L^{p'}(u_1), \ldots, L^{p'}(u_N))_{(\alpha,\beta),p';K} = L^{p'}(\hat{w}_{\alpha,\beta}^{-p'/p}).$$

Finally, duality formula (4) implies that

$$\begin{aligned}\overline{L^p(w)}_{(\alpha,\beta),p;J} &= ((L^{p'}(u_1), \ldots, L^{p'}(u_N))_{(\alpha,\beta),p';K})' = L^p(\hat{w}_{\alpha,\beta}) \\ &= \overline{L^p(w)}_{(\alpha,\beta),p;K}. \qquad \square\end{aligned}$$

5. Interpolation of N-tuples of classical Lorentz spaces

In this last section we work with classical Lorentz spaces, so we take $\Omega = (0, \infty)$ with the usual Lebesgue measure. Let $1 \leqslant p < \infty$ and let $\phi(x) > 0$ be an increasing concave function on $\mathbb{R}^+ = (0, \infty)$ such that $\phi(0+) = 0$. The classical Lorentz space Λ_ϕ^p (see [24,23]) is defined as the collection of all those measurable functions on $\mathbb{R}^+$ such that

$$\|f\|_{\Lambda_\phi^p} = \left(\frac{1}{p}\int_0^\infty f^*(x)^p \, d\phi(x)\right)^{1/p} < \infty.$$

If ψ is only a quasi-concave function with $\psi(0+) = 0$ and ϕ is its smallest concave majorant (so $\psi \simeq \phi$), we put $\Lambda_\psi = \Lambda_\phi$. The case $p = 1$ was already considered in Example 2.2. For our purposes, it will be useful to look at the norm in the form (see [7, Theorem 2.1])

$$\|f\|_{\Lambda_\phi^p} = \left(\int_0^\infty y^{p-1}\phi(\mu_f(y)) \, dy\right)^{1/p}.$$

Then $\|f\|_{\Lambda_\phi^p}^p = \int_0^\infty y^{p-1}\|\chi_{\{|f|>y\}}\|_{\Lambda_\phi} \, dy$, and so $\Lambda_\phi^p = \Lambda^p(\Lambda_\phi)$.

In what follows $\overline{\Lambda_\phi^p}$ denotes the N-tuple of classical Lorentz spaces $(\Lambda_{\phi_1}^p, \ldots, \Lambda_{\phi_N}^p)$. If $p = 1$, we simply write $\overline{\Lambda_\phi}$.

Lemma 5.1. *Let $A \subset (0, \infty)$ be a measurable set with Lebesgue measure $|A|$, then*

$$K(t, s; \chi_A; \overline{\Lambda_\phi}) \simeq \min_{1 \leqslant j \leqslant N}(t^{x_j}s^{y_j}\phi_j(|A|)).$$

Proof. Let $|A| = r$, since $(\chi_A)^* = \chi_{(0,r)}$ and $\sum(\overline{\Lambda_\phi})$ is invariant under rearrangement,

$$K(t, s; \chi_A; \overline{\Lambda_\phi}) = K(t, s; \chi_{(0,r)}; \overline{\Lambda_\phi}).$$

Given any $c > 1$, let $\chi_{(0,r)} = \sum_{j=1}^N f_j$ with $f_j \geqslant 0$ and $\sum_{j=1}^N t^{x_j} s^{y_j} \|f_j\|_{\Lambda_{\phi_j}} \leqslant cK(t, s; f; \overline{\Lambda_\phi})$. Then

$$\int_0^t \chi_{(0,r)}(x)\, dx = \int_0^t \left(\sum_{j=1}^N f_j\right)^*(x)\, dx \leqslant \sum_{j=1}^N \int_0^t f_j^*(x)\, dx \quad (t > 0).$$

According to Bennett and Sharpley [3], Theorem III.2.10 and Remark III.7.6, there is an operator $T : \Lambda_{\phi_j} \to \Lambda_{\phi_j}$ such that $\|T\|_{\mathcal{L}(\Lambda_{\phi_j} \to \Lambda_{\phi_j})} \leqslant 1$ and $\chi_{(0,r)} = \sum_{j=1}^N Tf_j^*$ with Tf_j^* decreasing. That means that there are $0 \leqslant c_j \leqslant 1$ such that $\sum_{j=1}^N c_j = 1$ and $Tf_j^* = c_j \chi_{(0,r)}$. Whence

$$\begin{aligned}\sum_{j=1}^N t^{x_j} s^{y_j} c_j \phi_j(r) &= \sum_{j=1}^N t^{x_j} s^{y_j} \|Tf_j^*\|_{\Lambda_{\phi_j}} \leqslant \sum_{j=1}^N t^{x_j} s^{y_j} \|f_j^*\|_{\Lambda_{\phi_j}} \\ &= \sum_{j=1}^N t^{x_j} s^{y_j} \|f_j\|_{\Lambda_{\phi_j}} \leqslant cK(t, s; \chi_{(0,r)}; \overline{\Lambda_\phi}).\end{aligned}$$

Thus

$$K(t, s; \chi_{(0,r)}; \overline{\Lambda_\phi}) \simeq \inf\left\{\sum_{j=1}^N t^{x_j} s^{y_j} c_j \phi_j(r) : 0 \leqslant c_j \leqslant 1, \sum_{j=1}^N c_j = 1\right\}.$$

This implies that

$$K(t, s; \chi_A; \overline{\Lambda_\phi}) \simeq \min_{1 \leqslant j \leqslant N} (t^{x_j} s^{y_j} \phi_j(|A|)). \qquad \square$$

Once we have the K-functional for $\overline{\Lambda_\phi}$ we can proceed as in Theorem 4.1 to determine the K-spaces generated by $\overline{\Lambda_\phi^p}$. The functions $\pi_{\alpha,\beta}$, $\check{\phi}_{\alpha,\beta}$, $\hat{\phi}_{\alpha,\beta}$ are defined as in Section 4. Note that properties of functions ϕ_j yield that $\pi_{\alpha,\beta}$ is also an increasing concave function with $\pi_{\alpha,\beta}(0+) = 0$.

Theorem 5.1. *Let* $\overline{\Lambda_\phi^p} = (\Lambda_{\phi_1}^p, \ldots, \Lambda_{\phi_N}^p)$, $1 \leqslant p < \infty$ *and let* $(\alpha, \beta) \in Int\ \Pi$. *Then*

(i) $(\overline{\Lambda_\phi^p})_{(\alpha,\beta),p;K} = \Lambda_{\pi_{\alpha,\beta}}^p$,

(ii) *If* (α, β) *does not lie on any diagonal of* Π, *then* $(\overline{\Lambda_\phi^p})_{(\alpha,\beta),p;K} = \Lambda_{\check{\phi}_{\alpha,\beta}}^p$.

Proof. (i) Using that $\Lambda_{\phi_j}^p = \Lambda^p(\Lambda_{\phi_j})$, it follows by Corollary 3.1 that

$$(\overline{\Lambda_\phi^p})_{(\alpha,\beta),p;K} = \Lambda^p((\overline{\Lambda_\phi})_{(\alpha,\beta),1;K}).$$

By Lemma 5.1

$$\|\chi_{\{|f|>y\}}\|_{\overline{(\Lambda_\phi)}_{(\alpha,\beta),1;K}} \simeq \int_0^\infty \int_0^\infty \min_{1\leqslant j\leqslant N}\{t^{x_j-\alpha}s^{y_j-\beta}\phi_j(\mu_f(y))\}\,\frac{dt}{t}\,\frac{ds}{s}.$$

Therefore

$$\begin{aligned}\|f\|^p_{\Lambda^p((\overline{\Lambda_\phi})_{(\alpha,\beta),1;K})} &= \int_0^\infty y^{p-1}\|\chi_{\{|f|>y\}}\|_{(\overline{\Lambda_\phi})_{(\alpha,\beta),1;K}}\,dy \\ &\simeq \int_0^\infty y^{p-1}\int_0^\infty\int_0^\infty \min_{1\leqslant j\leqslant N}\{t^{x_j-\alpha}s^{y_j-\beta}\phi_j(\mu_f(y))\} \\ &\quad\times\frac{dt}{t}\,\frac{ds}{s}\,dy \\ &= \int_0^\infty y^{p-1}\pi_{\alpha,\beta}(\mu_f(y))\,dy = \|f\|^p_{\Lambda^p_{\pi_{\alpha,\beta}}}.\end{aligned}$$

(ii) If (α,β) does not lie on any diagonal of Π, then we know from Theorem 4.1/(ii) that $\pi_{\alpha,\beta}\simeq\check{\phi}_{\alpha,\beta}$, which completes the proof. □

Next we turn our attention to the J-method. We start with the case $q=1$.

Theorem 5.2. *Let* $\overline{\Lambda_\phi}=(\Lambda_{\phi_1},\ldots,\Lambda_{\phi_N})$ *and* $(\alpha,\beta)\in \mathit{Int}\,\Pi$. *Then* $(\overline{\Lambda_\phi})_{(\alpha,\beta),1;J}=\Lambda_{\hat{\phi}_{\alpha,\beta}}$.

Proof. For $\{i,j,k\}\in\mathcal{P}_{\alpha,\beta}$, we put $L_{\{i,j,k\}}=(\Lambda_{\phi_i},\Lambda_{\phi_j},\Lambda_{\phi_k})_{(\alpha,\beta),1;J}$ if (α,β) belongs to the interior of the triangle with vertices P_i, P_j, P_k, and we put

$$L_{\{i,j,k\}}=\begin{cases}(\Lambda_{\phi_i},\Lambda_{\phi_j})_{\theta_{i,j},1} & \text{if } (\alpha,\beta)=(1-\theta_{i,j})P_i+\theta_{i,j}P_j,\\ (\Lambda_{\phi_i},\Lambda_{\phi_k})_{\theta_{i,k},1} & \text{if } (\alpha,\beta)=(1-\theta_{i,k})P_i+\theta_{i,k}P_k,\\ (\Lambda_{\phi_j},\Lambda_{\phi_k})_{\theta_{j,k},1} & \text{if } (\alpha,\beta)=(1-\theta_{j,k})P_j+\theta_{j,k}P_k\end{cases}$$

if (α,β) belongs to any edge of the triangle. By Lemmas 2.1 and 2.2

$$(\overline{\Lambda_\phi})_{(\alpha,\beta),1;J}\hookrightarrow\bigcap_{\{i,j,k\}\in\mathcal{P}_{\alpha,\beta}}L_{\{i,j,k\}}.$$

According to Asekritova and Krugljak [1], Theorem 1, K- and J-methods coincide for triples of Banach function spaces. Hence, using Theorem 5.1, we get $(\Lambda_{\phi_i},\Lambda_{\phi_j},\Lambda_{\phi_k})_{(\alpha,\beta),1;J}=\Lambda_{\phi_i^{c_i}\phi_j^{c_j}\phi_k^{c_k}}$, where (c_i,c_j,c_k) are the barycentric coordinates of (α,β) with respect to P_i, P_j, P_k. On the other hand, by Cerdà et al. [8], Theorem 10, $(\Lambda_{\phi_i},\Lambda_{\phi_j})_{\theta,1}=\Lambda_{\phi_i^{1-\theta}\phi_j^{\theta}}$. Whence, using that $\Lambda_{\phi_i}\cap\Lambda_{\phi_j}=\Lambda_{\max(\phi_i,\phi_j)}$, we conclude that

$$(\overline{\Lambda_\phi})_{(\alpha,\beta),1;J}\hookrightarrow\bigcap_{\{i,j,k\}\in\mathcal{P}_{\alpha,\beta}}L_{\{i,j,k\}}=\Lambda_{\hat{\phi}_{\alpha,\beta}}.$$

To establish the converse inclusion, note that the spaces Λ_{ϕ_j} are rearrangement invariant and $(\overline{\Lambda_\phi})_{(\alpha,\beta),1;J}$ is an exact interpolation space with respect to the N-tuple $\overline{\Lambda_\phi}$. Hence, by Bennett and Sharpley [3], Theorem III.2.12, we obtain that $(\overline{\Lambda_\phi})_{(\alpha,\beta),1;J}$ is rearrangement invariant. Let ψ be its fundamental function. By Cobos and Fernández-Martinez [10], Theorem 1.3

$$\begin{aligned}\psi(r) &= \|\chi_{(0,r)}\|_{(\Lambda_\phi)_{(\alpha,\beta),1;J}} \\ &\precsim \max_{\{i,j,k\}\in\mathcal{P}_{\alpha,\beta}} \{\|\chi_{(0,r)}\|^{c_i}_{\Lambda_{\phi_i}} \|\chi_{(0,r)}\|^{c_j}_{\Lambda_{\phi_j}} \|\chi_{(0,r)}\|^{c_k}_{\Lambda_{\phi_k}}\} \leqslant \hat{\phi}_{\alpha,\beta}(r).\end{aligned}$$

This implies that $\Lambda_{\hat{\phi}_{\alpha,\beta}} \hookrightarrow \Lambda_\psi \hookrightarrow (\overline{\Lambda_\phi})_{(\alpha,\beta),1;J}$ and completes the proof. □

We finish the paper by studying the case $1 < p < \infty$. First note that for any Banach N-tuple $\bar{A} = (A_1, \dots, A_N)$ and any $0 < \lambda \neq 1$ the space $\bar{A}_{(\alpha,\beta),q;J}$ is formed by all elements $a \in \sum(\bar{A})$ which can be represented as $a = \sum_{(m,n)\in\mathbb{Z}^2} u_{m,n}$ with $u_{m,n} \in \Delta(\bar{A})$ and $\left(\sum_{(m,n)\in\mathbb{Z}^2} (\lambda^{-\alpha m-\beta n} J(\lambda^m, \lambda^n; u_{m,n}))^q\right)^{1/q} < \infty$. Moreover $\|\cdot\|_{(\alpha,\beta),q;J}$ is equivalent to

$$\|a\|^{\sim}_{(\alpha,\beta),q;J} = \inf\left\{\left(\sum_{(m,n)\in\mathbb{Z}^2} (\lambda^{-\alpha m-\beta n} J(\lambda^m, \lambda^n; u_{m,n}))^q\right)^{1/q} : a = \sum_{(m,n)\in\mathbb{Z}^2} u_{m,n}\right\}.$$

We shall need the following auxiliary result.

Lemma 5.2. *Let $\bar{X} = (X_1, \dots, X_N)$ be a Banach N-tuple of r.i. spaces on $(0,\infty)$ and, for $1 < r < \infty$, let $\bar{X}^{(r)} = (X_1^{(r)}, \dots, X_N^{(r)})$ be the N-tuple formed by the convexifications. Then*

$$(\bar{X}_{(\alpha,\beta),1;J})^{(r)} \hookrightarrow (\bar{X}^{(r)})_{(\alpha,\beta),r;J}.$$

Proof. Given any $f \in (\bar{X}_{(\alpha,\beta),1;J})^{(r)}$ we can find a representation $|f|^r = \sum_{(m,n)\in\mathbb{Z}^2} u_{m,n}$ with

$$\left(\sum_{(m,n)\in\mathbb{Z}^2} 2^{-\alpha m-\beta n} J(2^m, 2^n; u_{m,n}; \bar{X})\right)^{1/r} \leqslant c\| |f|^r \|^{1/r}_{\bar{X}_{(\alpha,\beta),1;J}}.$$

Since $J(2^m, 2^n; u_{m,n}; \bar{X}) = J(2^{\frac{m}{r}}, 2^{\frac{n}{r}}; |u_{m,n}|^{1/r}; \bar{X}^{(r)})^r$, we can write the above inequality as

$$\left(\sum_{(m,n)\in\mathbb{Z}^2} (2^{-\frac{\alpha m}{r}-\frac{\beta n}{r}} J(2^{\frac{m}{r}}, 2^{\frac{n}{r}}; |u_{m,n}|^{1/r}; \bar{X}^{(r)}))^r\right)^{1/r} \leqslant c\|f\|_{(\bar{X}_{(\alpha,\beta),1;J})^{(r)}}.$$

Putting $\lambda = 2^{1/r}$ and $v_{m,n} = |u_{m,n}|^{1/r}$, we get

$$\left(\sum_{(m,n)\in\mathbb{Z}^2} (\lambda^{-\alpha m-\beta n} J(\lambda^m, \lambda^n; v_{m,n}; \bar{X}^{(r)}))^r\right)^{1/r} \leqslant c\|f\|_{(\bar{X}_{(\alpha,\beta),1;J})^{(r)}}.$$

This implies that $g = \sum_{(m,n)\in\mathbb{Z}^2} v_{m,n}$ belongs to $(\bar{X}^{(r)})_{(\alpha,\beta),r;J}$ with $\|g\|_{(\bar{X}^{(r)})_{(\alpha,\beta),r;J}} \leqslant c_1\|f\|_{(\bar{X}_{(\alpha,\beta),1;J})^{(r)}}$. Since

$$|f| = \left(\sum_{(m,n)\in\mathbb{Z}^2} u_{m,n}\right)^{1/r} \leqslant \sum_{(m,n)\in\mathbb{Z}^2} |u_{m,n}|^{1/r} = \sum_{(m,n)\in\mathbb{Z}^2} v_{m,n} = g$$

and $(\bar{X}^{(r)})_{(\alpha,\beta),r;J}$ is rearrangement invariant (the same argument as for $\overline{\Lambda^1(w)}_{(\alpha,\beta),1;J}$ applies), we conclude that f belong to $(\bar{X}^{(r)})_{(\alpha,\beta),r;J}$ with $\|f\|_{(\bar{X}^{(r)})_{(\alpha,\beta),r;J}} \leqslant c_1\|f\|_{(\bar{X}_{(\alpha,\beta),1;J})^{(r)}}$. $\square$

Now we are ready for the description of J-spaces when $1 < p < \infty$.

Theorem 5.3. *Let* $\overline{\Lambda^p_\phi} = (\Lambda^p_{\phi_1}, \dots, \Lambda^p_{\phi_N})$, *let* $1 < p < \infty$ *and let* $(\alpha,\beta) \in \mathrm{Int}\ \Pi$ *such that* (α,β) *does not lie on any diagonal of* Π, *then*

$$(\overline{\Lambda^p_\phi})_{(\alpha,\beta),p;J} = \Lambda^p_{\hat{\phi}_{\alpha,\beta}}.$$

Proof. Since $\Lambda^p_{\phi_j} = (\Lambda_{\phi_j})^{(p)}$ $(j = 1, \dots, N)$, by the previous lemma and Theorem 5.2 we get that

$$\Lambda^p_{\hat{\phi}_{\alpha,\beta}} = (\Lambda_{\hat{\phi}_{\alpha,\beta}})^{(p)} = ((\overline{\Lambda_\phi})_{(\alpha,\beta),1;J})^{(p)} \hookrightarrow (\overline{\Lambda^p_\phi})_{(\alpha,\beta),p;J}.$$

To establish the converse embedding we can proceed as in the first part of Theorem 5.2 because, by the assumption on (α,β), for any $\{i,j,k\} \in \mathcal{P}_{\alpha,\beta}$ the point (α,β) belongs to the interior of the triangle with vertices P_i, P_j, P_k. $\square$

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